Phase diagram and critical exponents of a Potts gauge glass

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(Received 14 June 2001; published 16 January 2002)

The two-dimensional q-state Potts model is subjected to a Z_q symmetric disorder that allows for the existence of a Nishimori line. At $q=2$, this model coincides with the $\pm J$ random-bond Ising model. For $q>2$, apart from the usual pure- and zero-temperature fixed points, the ferro/paramagnetic phase boundary is controlled by *two* critical fixed points: a weak disorder point, whose universality class is that of the *ferromagnetic* bond-disordered Potts model, and a strong disorder point which generalizes the usual Nishimori point. We numerically study the case $q=3$, tracing out the phase diagram and precisely determining the critical exponents. The universality class of the Nishimori point is inconsistent with percolation on Potts clusters.

DOI: 10.1103/PhysRevE.65.026113 PACS number(s): 75.50.Lk, 05.50.+q, 64.60.Fr

During the last decade, the study of disordered systems has attracted much interest. This is true in particular in two dimensions, where the possible types of critical behavior for the corresponding pure models can be classified using conformal field theory [1]. Recently, similar classification issues for disordered models have been addressed through the study of various random matrix ensembles [2], but many fundamental questions remain open.

An important category of two-dimensional (2D) disordered systems is given by models where the disorder couples to the local energy density. Two paradigmatic members of this class are the $\pm J$ random-bond Ising model, and the q-state ferromagnetic random-bond Potts model. The model to be studied in the present paper can be thought of as an interpolation between these two members; we shall therefore begin by recalling some of their basic properties.

The random-bond Ising model (RBIM) is defined by the energy functional

$$\mathcal{H}_{\text{Ising}}=-\sum_{\langle i,j\rangle} J_{ij}\delta(S_i,S_j), \tag{1}$$

where the sum is over the edges of the square lattice, $S_i=\pm 1$ are Ising spins, and $\delta(.,.)$ is the Kronecker delta function. The random bonds take the values $J_{ij}=\pm 1$ with probability p, respectively, $1-p$. The salient feature of this model is that it marries disorder with frustration, leading to the possibility of spin glass order.

Its phase diagram is generally believed to be as in Fig. 1(a) [3]. The boundary (FP) between the ferromagnetic and the paramagnetic phases is controlled by three fixed points. The attractive fixed points at either end of the phase boundary are, respectively, the critical point of the pure Ising model and a zero-temperature fixed point. Between these two we find the multicritical point N, intersecting the so-called Nishimori line $e^{\beta}=(1-p)/p$ [4]. On this line, the replicated version of the model possesses a local Z_2 gauge symmetry that, among other things, allows for exactly computing the internal energy and for establishing the pairwise equality of correlation functions

$$[\langle S_{i_1}S_{i_2}\cdots S_{i_k}\rangle^{2n-1}]=[\langle S_{i_1}S_{i_2}\cdots S_{i_k}\rangle^{2n}], \tag{2}$$

where $\langle\cdots\rangle$ denotes the thermal and $[\cdots]$ the disorder average. Since the Nishimori line is also invariant under renormalization group (RG) transformations [3], its intersection N with the FP boundary must be a fixed point. However, the widespread belief that the corresponding universality class is of the percolation type has recently been refuted on the basis of numerical evidence [5].

The other model of special interest to us is the random-bond Potts model (RBPM), which is also defined by Eq. (1), except that the spins now take q different values, $S_i=1,2,\ldots,q$. The most well-studied case is that of ferromagnetic bonds $J_1,J_2>0$, each taken with probability 1/2, and with $R\equiv J_2/J_1\geqslant 1$ adjusting the disorder strength.

In contradistinction to the Nishimori point, the fixed point of this model is situated at weak disorder. For $q>2$ the disorder is relevant [6], and the corresponding line of fixed points tends to the one of the pure Ising model in the limit $q\to 2$. As a consequence, the critical exponents can be computed perturbatively in a $(q-2)$ expansion [7]. According to the RG picture, for $q>2$, any small amount of disorder should induce a flow towards the random fixed point. That this is also true for $q>4$, where the phase transition in the pure model is of the first order, is the content of the Aizenman-Wehr theorem [8].

In this paper, we shall consider the model

$$\mathcal{H}=-\sum_{\langle i,j\rangle}\delta^{(q)}(S_i-S_j+J_{ij}), \tag{3}$$

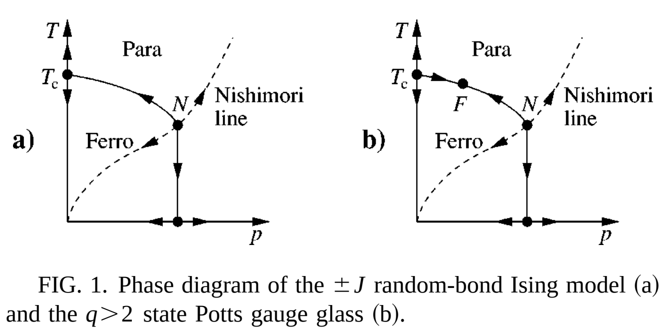

FIG. 1. Phase diagram of the $\pm J$ random-bond Ising model (a) and the $q>2$ state Potts gauge glass (b).

where $S_i = 1,2,\ldots,q$, and $\delta^{(q)}(x)=1$ if $x=0 \bmod q$ and zero otherwise. The randomness now takes the form of a local "twist" J_{ij}, which is clearly a more severe type of disorder than simple bond randomness. The variables J_{ij} are taken from the distribution

$$P(J_{ij}) = [1-(q-1)p]\delta(J_{ij}) + p\sum_{J=1}^{q-1}\delta(J_{ij}-J), \quad (4)$$

with $0 \leqslant p \leqslant 1/(q-1)$ controlling the strength of the randomness. We shall refer to this model, which was originally introduced in Ref. [9], as the Potts gauge glass (PGG). The particular form of the randomness ensures the existence of a Nishimori line (see below). For $q=2$, the PGG reduces to the RBIM, and for $p=1/q$ it was studied analytically in [10]. It is also connected to the RBPM: To wit, when $q>2$, the pure Potts model ($p=0$) should be *unstable* to a small amount of randomness, meaning that the RG flow cannot be as indicated on Fig. 1(a). Instead, we are forced to assume the existence of a new fixed point *F*, intermediary between the pure model and the Nishimori point [see Fig. 1(b)]. But whenever $(q-2)$, and hence the value of p at *F*, is sufficiently small, frustration effects are negligible, and we should flow to the *same* random fixed point as in the RBPM. For reasons of continuity, we expect this argument to hold true also for higher values of q.

The expression of the Nishimori line was obtained in Ref. [9], but since our notation is slightly different we shall repeat the argument here. We first re-express the disorder distribution as

$$P(J_{ij}) = p e^{K\delta^{(q)}(J_{ij})} \quad \text{with} \quad K = \log[1/p-(q-1)]. \quad (5)$$

Consider then the disorder averaged internal energy

$$E = \mathcal{N}\sum_{\{J_{ij}\}}\left(\prod_{\langle ij\rangle} e^{K\delta^{(q)}(J_{ij})}\right)\frac{\sum_{\{S_i\}}\delta^{(q)}(S_i-S_j+J_{ij})e^{-\beta\mathcal{H}}}{\sum_{\{S_i\}}e^{-\beta\mathcal{H}}}, \quad (6)$$

where $\mathcal{N} = -1/(q-1+e^K)^{2N}$, N being the number of sites of the square lattice. $\mathcal{H}$ is then invariant under the gauge transformation $S_i \to S_i - \sigma_i, J_{ij} \to J_{ij} + \sigma_i - \sigma_j$, though $P(J_{ij})$ is not. Still, E is invariant since we sum over all configurations of the disorder. Then, averaging over all the possible gauge transformations, we get

$$E = \mathcal{N}q^{-N}\sum_{\{J_{ij}\}}\sum_{\{\sigma_i\}}\left(\prod_{\langle ij\rangle} e^{K\delta^{(q)}(J_{ij}+\sigma_i-\sigma_j)}\right) \times \frac{\sum_{\{S_i\}}\delta^{(q)}(S_i-S_j+J_{ij})e^{-\beta\mathcal{H}}}{\sum_{\{S_i\}}\prod_{\langle i'j'\rangle}e^{\beta\delta^{(q)}(S_{i'}-S_{j'}+J_{i'j'})}}. \quad (7)$$

Imposing $K=\beta$, there is a remarkable simplification

$$E = \mathcal{N}q^{-N}\sum_{\{J_{ij}\}}\sum_{\{S_i\}}\delta^{(q)}(S_i-S_j+J_{ij})e^{-\beta\mathcal{H}} = \mathcal{N}q^{-N}\frac{\partial}{\partial\beta}\sum_{\{S_i\}}(e^\beta+q-1)^{2N} = \frac{-2Ne^\beta}{q-1+e^\beta}. \quad (8)$$

Thus, E is regular, and Eq. (5) with $K=\beta$ defines the generalized Nishimori line.

Normalized two-point functions are defined by

$$\langle S_i S_j\rangle = (q-1)^{-1}(q\langle\delta^{(q)}(S_i-S_j)\rangle - 1). \quad (9)$$

Let us now recall how Eq. (2) can be derived for $q=2$. We consider $n=1$ for simplicity. Using the trivial identities $\delta^{(q)}(\Delta S-\Delta\sigma) = \Sigma_{l=0}^{q-1}\delta^{(q)}(\Delta S - l)\delta^{(q)}(\Delta\sigma + l)$ and $\Sigma_{l=0}^{q-1}\delta^{(q)}(\Delta S - l)=1$, one readily establishes that $2\delta^{(2)}(\Delta S-\Delta\sigma)-1 = (2\delta^{(2)}(\Delta S)-1)(2\delta^{(2)}(\Delta\sigma)-1)$ and, using the same gauge transformation as before,

$$[2\langle\delta^{(2)}(S_i-S_j)\rangle - 1] = [(2\langle\delta^{(2)}(S_i-S_j)\rangle-1)^2]. \quad (10)$$

This relies crucially on the fact that the above trivial identities generate only two terms, and for general q we do not expect simple relations such as Eq. (2).[1]

We now turn to our numerical results. Random transfer matrices in the Fortuin-Kasteleyn (FK) representation [11] have been a very powerful tool for studying the RBPM [12]. Unfortunately, the random twist variables J_{ij} present in Eq. (3) complicate the definition of the FK clusters: only those clusters are allowed for which $\Sigma_\gamma J_{ij} = 0 \bmod q$ for any path γ within the cluster [13]. It is not obvious how this constraint can be generalized to *real* values of q, and even for integer q keeping track of the necessary local information would greatly increase the number of basis states needed. We have therefore found it more convenient to write the transfer matrices directly in the spin basis. We work at $q=3$ throughout, but expect our conclusions to extend to arbitrary $q>2$.

As we have shown in an earlier publication [14], the phase diagram can be traced out by investigating the effective central charge. To this end we have computed the free energy $f_L^{(p)} = \ln Z^{(p)}/LM$ on strips of various widths L and practically infinite length, $M=10^5$. The (effective) central charge c can then be obtained as the universal coefficient of the finite-size correction to the free energy for periodic boundary conditions [15]

$$f_L^{(p)} = f_\infty^{(p)} + \frac{c\pi}{6L^2} + \cdots. \quad (11)$$

According to Zamolodchikov's c theorem [16], here applied to a nonunitary theory, the effective central charge *increases* along the RG flows and coincides with the (true) central

[1] A simple relation for a chiral-type correlator was established in Ref. [9], but in terms of Eq. (9) this does not lead to degeneracy in the multiscaling spectrum.

TABLE I. Parametrization of the ferro/paramagnetic phase boundary.

	$L=3,4$		$L=4,5$		$L=2,3,4$		$L=3,4,5$	
p	β	c	β	c	β	c	β	c
0.01	1.0521(5)	0.76825(3)	1.0520(5)	0.78074(9)	1.0520(5)	0.79460(6)	1.0520(5)	0.7987(3)
0.02	1.1061(5)	0.76874(6)	1.1061(5)	0.7815(2)	1.1061(5)	0.7953(1)	1.1061(5)	0.7998(6)
0.03	1.1692(5)	0.76907(9)	1.1691(5)	0.7816(2)	1.1692(5)	0.7957(2)	1.1691(5)	0.7997(6)
0.04	1.244(1)	0.7685(1)	1.245(1)	0.7822(7)	1.244(1)	0.7951(3)	1.245(1)	0.8020(17)
0.05	1.336(1)	0.7663(3)	1.337(1)	0.7799(9)	1.336(1)	0.7925(6)	1.338(1)	0.7995(25)
0.06	1.453(2)	0.7620(3)	1.456(2)	0.7739(11)	1.454(2)	0.7882(6)	1.456(2)	0.7911(30)

charge at the fixed points. The FP boundary (cf. Fig. 1) can be traced by identifying the maximum of c as a function of T, for various fixed values of p.

Since the randomness is strong, and since the fits to Eq. (11) must be based on at least two different sizes L to eliminate the nonuniversal quantity $f_{\infty}^{(p)}$, we have taken several precautions in order to obtain small error bars on the $f_L^{(p)}$. First, for any fixed value of p, we use the *same* realization of the disorder for the computations at different values of T. Second, for each strip of length $M=10^5$, we work in a canonical ensemble, meaning that disorder realizations for which the fraction of bonds $J_{ij}=J$ does not *exactly* equal p for each $J=1,2,\dots,q-1$ are discarded. Third, for each strip we average $f_L^{(p)}$ over up to 10^5 independent realizations.

In Table I we show the resulting values of c and the inverse temperature $\beta=1/T$ at the FP boundary. The two-point fits are based directly on Eq. (11), while the three-point fits include an additional nonuniversal $1/L^4$ correction [12]. The existence of an attractive fixed point at $p\sim 0.04$ with a central charge slightly larger than $c_{\rm pure}=4/5$, characterizing the pure three-state Potts model, is brought out very clearly.

The reader may wonder why data for such small system sizes can possible give any reliable information about the thermodynamic limit. Comparison with the pure model ($p=0$) shows however that in particular, the three-point fits converge very rapidly towards the exact result: $c_{3,4}=0.768\,03$, $c_{4,5}=0.780\,43$, $c_{2,3,4}=0.794\,31$, $c_{3,4,5}=0.79831$ [17]. We have extrapolated the data at the fixed point F by assuming that for each fit, the relative deviation from the infinite-size result is the same as in the pure model. In this way, we arrive at the final result

$$c_{\rm F}=0.802\,5(10), \tag{12}$$

which compares favorably with the perturbative result $c_{\rm pert}=4013/5000+\mathcal{O}(q-2)^5\approx 0.8026$ [7] for the *ferromagnetic* RBPM.

To numerically locate the Nishimori point, we measure $c_{\rm eff}$ along the Nishimori line. Since in this case p is a function of β [see Eq. (5) with $K=\beta$] we can no longer work in the canonical ensemble of disorder realizations. Accordingly our error bars are larger. It is however a big advantage to know the exact parametrization of the Nishimori line, since otherwise we would have had to scan a two-dimensional manifold of parameter values [18].

From the data in Table II we conclude that the fixed point N is located at $p_N=0.078\,5(10)$. Using the same extrapolation procedure as above, we also estimate

$$c_{\rm N}=0.756(5). \tag{13}$$

This is in remarkable agreement with the value of the central charge for the percolation limit in the RBPM: $c=5\sqrt{3}\ln q/4\pi\approx 0.7571$ [12]. Below, we shall return to the question whether the Nishimori point is ''just'' percolation.

We have also measured magnetic multiscaling exponents η_n, which, after a conformal mapping [5], are defined on the semi-infinite cylinder of circumference L, with $x\in[1,L]$ and $y\in[-\infty,+\infty]$, by

$$[\langle S(x_1,y)S(x_2,y)\rangle^n]\propto\left[\sin\left(\frac{\pi(x_2-x_1)}{L}\right)L\right]^{-\eta_n}. \tag{14}$$

For a pure system, $\eta_n=n\times\eta$, while for percolation over Potts clusters all η_n coincide. The principal goal here is to establish the nontrivial multiscaling at N, rather than to determine the η_k with extraordinary precision. The largest system size employed was $L=12$, and we approximate the semi-infinite cylinder by taking a length of $M=400L$. All runs were averaged over 10^3 disorder configurations.

In Fig. 2, we show effective values of $\eta_1(L)$ along the Nishimori line, for various p close to p_N. These values were obtained by fitting data for all $x_2-x_1=1,\dots,L/2$ to Eq. (14); to judge the systematic error due to the inclusion of the

TABLE II. Effective central charge along the Nishimori line.

p	$L=3,4$	$L=4,5$	$L=2,3,4$	$L=3,4,5$
0.077	0.7208(4)	0.7284(22)	0.7374(8)	0.739(6)
0.078	0.7212(5)	0.7346(27)	0.7374(10)	0.754(7)
0.079	0.7218(5)	0.7316(22)	0.7386(11)	0.746(6)
0.080	0.7213(7)	0.7292(24)	0.7379(14)	0.741(6)

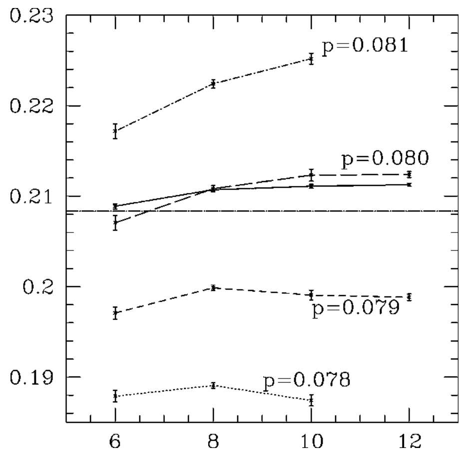

FIG. 2. η_1 extracted from Eq. (14) with $\Delta x = 1, \ldots, L$. We also show the corresponding fit for percolation (full line) and the exact value $\eta_{\mathrm{perc}} = 5/24$.

smallest $\Delta x \equiv |x_2 - x_1|$ we also display a similar plot for ordinary percolation, where $\eta_{\mathrm{perc}} = 5/24 \simeq 0.2083$ is known exactly. At the fixed point, $\eta_1(L)$ must tend to a constant, and we conclude that $p_N = 0.079 - 0.080$ with $\eta_1 = 0.20 - 0.21$. Discarding the smallest Δx leads to consistent results, but with larger error bars.

Although our value of η_1 is consistent with percolation, this scenario can be excluded by considering higher moments. E.g., for $p = 0.080$ and $L = 12$ we obtain

$$\eta_1 = 0.212\,39(35) \qquad \eta_2 = 0.251\,92(39)$$

$$\eta_3 = 0.308\,24(47) \qquad \eta_4 = 0.337\,73(52), \tag{15}$$

the corresponding values for $p = 0.079$ being some 6% smaller.

Further evidence against percolation can be obtained by similarly considering the energy-energy correlations. In analogy with the RBIM case we associate this with a deviation from N along the *vertical* direction on Fig. 1(b). In this case, the exponents depend less on the precise value of p_N, but the finite-size corrections are larger. Extrapolating, we find a value of roughly $\eta_1^{\mathrm{e}} = 2.75 - 2.85$, rather close to the one obtained for the RBIM Nishimori point $\eta_1^{\mathrm{e}} = 2.83(2)$ using a similar fit [19], and significantly larger than the percolation value $\eta_{\mathrm{perc}}^{\mathrm{e}} = 2(2 - 1/\nu_{\mathrm{perc}}) = 5/2$. Discarding data with small Δx leads to larger error bars, but is still consistent with $\eta_1^{\mathrm{e}} \sim 2.85$. We have also verified that the energy correlations exhibit genuine multiscaling.

In conclusion, we have studied a q-state (Potts-like) generalization of the $\pm J$ random-bond Ising model that allows for the definition of a Nishimori line. Apart from a weak disorder fixed point that coincides with that of the well-studied random-bond Potts model, the model possesses a strong disorder point with multiscaling exponents different from those of percolation. The fixed-point structure is reminiscent of that found by Sørensen *et al.* [18] in the context of a $\pm J$-like Potts model, which does however not possess the gauge symmetry required for defining a Nishimori line. We believe that it would be interesting to study whether the critical points of these two models are indeed identical. Open questions concerning our model include the study of its zero-temperature limit, the possibility of reentrance, and of its behavior for $q > 4$. It would also be interesting to examine it using a supersymmetric approach.

We would like to thank J. Cardy, A. Honecker, and P. Pujol for useful discussions.

[1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984); J. Stat. Phys. **34**, 763 (1984).
[2] M. R. Zirnbauer, J. Math. Phys. **37**, 4986 (1996); A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[3] W. L. McMillan, Phys. Rev. B **29**, 4026 (1984); A. Georges and P. Le Doussal (unpublished); P. Le Doussal and A. B. Harris, Phys. Rev. Lett. **61**, 625 (1988); Phys. Rev. B **40**, 9249 (1989).
[4] H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981); J. Phys. Soc. Jpn. **55**, 3305 (1986); Y. Ozeki and H. Nishimori, J. Phys. A **26**, 3399 (1993).
[5] A. Honecker, M. Picco, and P. Pujol, Phys. Rev. Lett. **87**, 047201 (2001).
[6] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[7] A. W. W. Ludwig and J. L. Cardy, Nucl. Phys. B **285**, 687 (1987); A. W. W. Ludwig, *ibid.* **330**, 639 (1990); Vl. Dotsenko, M. Picco, and P. Pujol, *ibid.* **455**, 701 (1995).
[8] M. Aizenman and J. Wehr, Phys. Rev. Lett. **62**, 2503 (1989).
[9] H. Nishimori and M. J. Stephen, Phys. Rev. B **27**, 5644 (1983).
[10] Y. Y. Goldschmidt, J. Phys. A **22**, L157 (1989).
[11] P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jpn. **46** (Suppl.), 11 (1969).
[12] J. L. Jacobsen and J. L. Cardy, Nucl. Phys. B **515**, 701 (1998).
[13] M. Caselle, F. Gliozzi, and S. Necco, J. Phys. A **34**, 351 (2001).
[14] J. L. Jacobsen and M. Picco, Phys. Rev. E **61**, R13 (2000).
[15] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986); I. Affleck, *ibid.* **56**, 746 (1986).
[16] A. B. Zamolodchikov, Pis'ma Zh. Éksp. Teor. Fiz. **43**, 565 (1986) [JETP Lett. **43**, 730 (1986)].
[17] We have chosen to transfer along the (1,1) direction of the square lattice, since (at least for the pure model) this yields a faster convergence, and, more importantly, a monotonic one.
[18] E. S. Sørensen, M. J. P. Gingras, and D. A. Huse, Europhys. Lett. **44**, 504 (1998).
[19] A. Honecker, M. Picco, and P. Pujol (unpublished).